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New energetic and dynamic quantum effects originating from the breaking of time-reversal symmetry

Dieter Schuch

Institut für Theoretische Physik, J W Goethe-Universität, Robert-Mayer-Strasse 8-10, D-60054 Frankfurt am Main, Germany

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Abstract
The reduction of spatial symmetry can have qualitative effects for a given quantum system, such as splitting of energy levels that are degenerate in the unperturbed case. Can the breaking of temporal symmetry have a similar effect? Using a nonlinear quantum mechanical model to describe the effective interaction of simple, exactly solvable quantum systems with some dissipative environment, the time-symmetry-breaking effect of this interaction on the ground-state energy of these systems can be studied analytically. Also, in the case of dynamical properties such as tunnelling currents, the breaking of time-reversal symmetry can have quite unexpected effects. In general, a dissipative environment is assumed to inhibit the motion of a dynamical system, e.g., by reducing the frequency and amplitude of a harmonic oscillator. Our study shows, however, that under certain resonance-like conditions, the interaction with the environment can also create currents that are not present in the unperturbed system. These results follow directly from the analytical solutions of our model.

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1. Introduction

Classical Hamiltonian dynamics, as well as conventional quantum mechanics, are reversible, conservative theories, i.e. the direction of time does not matter and the energy of the system is a constant. These theories apply solely to isolated systems. Real systems, however, are usually in contact with some kind of environment—at the very least, when they are measured in an experiment, they are no longer isolated. It can be assumed—and this is the conventional point of view—that the interaction with the environment only causes some small *quantitative* changes in the behaviour of the system that can be described adequately by, e.g., perturbation theory. However, the interaction with some environment usually breaks the time-reversal

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symmetry of the dynamics of a system. Also, it is known, e.g., from ligand-field theory, that the breaking of spatial symmetry can have quite important *qualitative* consequences, e.g., splitting of degenerate energy levels. One question to be addressed in this paper is whether a similar energetic effect can also arise by breaking the temporal symmetry. However, it is not only the energetics of a quantum system that is influenced by an environment, it is the dynamics as well. For certain systems in contact with some dissipative environment, e.g., in the quantum Hall effect or high-T_C superconductivity, it is known that the change of an environmental parameter, for instance the temperature, can result in qualitative changes of the dynamics of the system, namely, in the creation or destruction of currents. The possibility of qualitative changes of the dynamics of dissipative quantum systems that can be described simply by introducing and varying an environmental parameter will also be addressed in this paper.

Since we are interested only in the system and the effect the environment has on it, but not in the details of the interaction or the environment, we choose a model that allows for the description of a system by a wavefunction that can always be normalized, even if it is in contact with some environment, and that possesses exact analytical solutions, thus allowing the qualtitative consequences of changes of parameters, etc to be seen immediately. This model is based on a nonlinear (NL) modification of the Schrödinger equation (SE) and can be traced back to the conventional system-plus-reservoir approach, which will be discussed in section 2. For the systems we are interested in, the NLSE possesses exact solutions in the form of Gaussian wave packets (WP). Therefore, in section 3, we give a short outline of the dynamics and energetics of these WPs and introduce the quantities that will be applied to describe and study these properties.

In section 4, the qualtitative changes in the dynamics and energetics of the WPs, caused by the breaking of time-reversal symmetry, will be discussed in detail for the damped free motion and the damped harmonic oscillator. Finally, in section 5, the results will be summarized and possible consequences discussed.

2. Physically equivalent models for the description of dissipative systems

A conventional way of describing a dissipative system with irreversible time-evolution is the system-plus-reservoir approach [1–3], where the system of interest is coupled to a large number of environmental degrees of freedom (e.g. harmonic oscillators), with subsequent elimination of these by some averaging processes. The whole, i.e. the system-plus-reservoir, is considered a closed Hamiltonian system (classically, as well as quantum mechanically). Due to the large number of environmental degrees of freedom that are involved and the necessary limits and averages that have to be taken, the calculations can become rather cumbersome and lose transparency. Other approaches, therefore, attempt, from the very beginning, to find a Hamiltonian description of the dissipative system alone, without taking the environment explicitly into account. The most well-known, and often-quoted, approach is that of Caldirola [4] and Kanai [5] which uses an explicitly time-dependent Hamiltonian of the form

$$H_{\mathrm{CK}} = \frac{1}{2m} \mathrm{e}^{-\gamma t} p_{\mathrm{CK}}^2 + \mathrm{e}^{\gamma t} V(x) \tag{1}$$

(with canonical momentum $p_{\mathrm{CK}} = p \exp(\gamma t)$, where p is the usual kinetic momentum $p = m\dot{x}$) which yields the proper equations of motion for a system exposed to a linear velocity-dependent frictional force. This model and its quantum analogue, obtained by canonical quantization, have been discussed extensively in the literature (see e.g. [6–12]). Although it was applied frequently for the description of dissipative systems, it was also criticized by various authors [13–15]. The most serious point of criticism raised against this model is that after quantization, this Hamiltonian leads to an apparent violation of Heisenberg's uncertainty principle. A puzzling situation arose when Sun and Yu [16] showed that the same Hamiltonian operator could be obtained by standard procedures, just starting from the above-mentioned conventional system-plus-reservoir approach and should, therefore, be physically equivalent to this. This puzzle was solved by showing that it is actually not the Hamiltonian operator that leads to the violation of the uncertainty relation, but the inappropriate treatment of the corresponding wavefunctions. Since the transition to H_{CK} involves, classically, a non-canonical and, quantum mechanically, a non-unitary transformation, the wavefunctions must also be transformed accordingly and have a different physical meaning in the Caldirola–Kanai model from their meaning in the physical system described by the SE (in the conservative case); for a detailed discussion see [17].

However, the problems of violation of the uncertainty principle and transformation of the wavefunction do not occur in our model which uses a nonlinear modification of the SE that describes the system alone, i.e. similar to the Caldirola–Kanai model but, in our case, the interpretation of the wavefunction and the quantities calculated with them (such as position and momentum uncertainties) are the same as for the usual SE. Moreover, it has been shown [17] that our NLSE can be transformed into the Caldirola–Kanai equation with the help of a well-defined transformation that establishes the physical equivalence of the two models and, thus, also the physical equivalence between our description using an NLSE and the corresponding system-plus-reservoir approach. An advantage of our approach, compared to that of the system-plus-reservoir, is that it immediately yields exact analytical solutions and it also does not suffer the shortcomings of several other approaches [18–20] also using NLSEs for the description of dissipative systems (see also [6]).

A short outline of how to obtain our NL model is given in the following. The investigations shall be restricted to one spatial dimension but extension to higher dimensions is straightforward. It has been shown [21–23] that the above-mentioned problems can be solved, or avoided, using an approach that starts by modifying the continuity equation

$$\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x} j(x,t) = \frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x}(\varrho v_-) = 0 \tag{2}$$

for the density $\varrho(x,t) = \Psi^*(x,t)\Psi(x,t)$ (where the convection current density $j(x,t)$ is defined as $j = \varrho v_- = (\frac{\hbar}{2m\mathrm{i}})(\Psi^* \frac{\partial}{\partial x}\Psi - \Psi \frac{\partial}{\partial x}\Psi^*)$) with the wavefunction $\Psi(x,t)$ being the solution of the corresponding time-dependent SE,

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\Psi_L(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi_L(x,t) = H_L \Psi_L(x,t). \tag{3}$$

Madelung [24] and Mrowka [25] had introduced a method of separating the real continuity equation (2) into two complex conjugate equations in order to rederive the conventional linear SE without applying Hamilton's form of classical mechanics, only Newton's form (that also allows for dissipative frictional forces). Our modification consists in adding a time-symmetry-breaking diffusion term to the continuity equation, thus arriving at the Fokker–Planck-type equation

$$\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x}(\varrho v_-) - D\frac{\partial^2}{\partial x^2}\varrho = 0. \tag{4}$$

The inclusion of a diffusion current density was later supported on group theoretical grounds; it was derived from the analysis of representations of the Diff($\mathbf{R}^3$) group by Doebner and Goldin [26]. However, it faces a problem when trying to follow Madelung and Mrowka's procedure, as separation of the term

$$-D\frac{\frac{\partial^2}{\partial x^2}\varrho}{\varrho} = -D\left[\frac{\frac{\partial^2}{\partial x^2}\Psi}{\Psi} + \frac{\frac{\partial^2}{\partial x^2}\Psi^*}{\Psi^*} + 2\left(\frac{\frac{\partial}{\partial x}\Psi}{\Psi}\right)\left(\frac{\frac{\partial}{\partial x}\Psi^*}{\Psi^*}\right)\right] \tag{5}$$

because of the last contribution on the rhs, is not generally possible.

Nevertheless, it could be shown that there are physically meaningful cases where fulfilment of an additional condition allows separation. The condition we use is

$$-D\frac{\frac{\partial^2}{\partial x^2}\varrho}{\varrho} = \gamma(\ln\varrho - \langle\ln\varrho\rangle) \tag{6}$$

(with $\langle\cdots\rangle = \int\Psi^*\cdots\Psi\,\mathrm{d}x$), where the second term on the rhs is necessary to allow for normalization of the wavefunctions. After separation, the choice of (6) for the diffusion term leads to the NLSE

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_{\mathrm{NL}}(x,t) &= \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + \gamma\frac{\hbar}{\mathrm{i}}(\ln\Psi_{\mathrm{NL}} - \langle\ln\Psi_{\mathrm{NL}}\rangle)\right]\Psi_{\mathrm{NL}}(x,t) \\ &= \left[H_L + \gamma\frac{\hbar}{\mathrm{i}}(\ln\Psi_{\mathrm{NL}} - \langle\ln\Psi_{\mathrm{NL}}\rangle)\right]\Psi_{\mathrm{NL}}(x,t). \end{aligned} \tag{7}$$

The logarithmic nonlinearity corresponds to a linear velocity-dependent frictional force, with friction coefficient γ, caused by the interaction with a dissipative environment that does not appear explicitly. Despite the imaginary contribution from the logarithmic term, the solutions of the NLSE are normalizable.

It should be mentioned that there are also other ways of treating the diffusion term in the Fokker–Planck equation (4) in order to obtain a corresponding complex SE. Doebner and Goldin, e.g., added half of the non-separable contribution on the rhs of equation (5) to each of the equations for Ψ and Ψ^*, also arriving at an NLSE [26], but having to pay the price of the equation for Ψ also containing explicitly Ψ^* and vice versa (for further details see [27]).

It also should be noted that nonlinear modifications of the SE have usually some problematic aspects, e.g. the fact that, in general, the superposition principle does not apply. However, it has been shown that in our model, e.g., Gaussian WP solutions of the log NLSE for the damped free motion can be constructed by a superposition of plane-wave type solutions that are similar to those in the undamped case [22]. This might be related to the property of our nonlinear equation being linearizable by a non-unitary transformation to yield the Caldirola–Kanai equation. Similarly, the above-mentioned Doebner–Goldin NLSE can also be linearized to yield the same Caldirola–Kanai equation, however, with the help of a nonlinear gauge transformation. (For further details see [28].)

It even seems possible to make a link to the usual description of dissipative quantum systems in the framework of a linear master equation for the density operator. Progress in this direction, starting on the wavefunction level from the Doebner–Goldin NLSE, has been achieved by Dodonov and Mizrahi [29, 30]. Connections between an NLSE and a master equation of Lindblad form have also been shown by Gisin [31]. Since our log NLSE can also be written in a form similar to that applied by Gisin and is, for certain constraints, identical with the Doebner–Goldin equation, the desired links should also be possible in our case. Work in this direction is in progress.

Under certain conditions, however, a description of dissipative systems in terms of an SE, rather than a master equation, is possible, but then, generally, fluctuations have to be considered which give rise to stochastic terms. In principle, the above-mentioned derivation of our NLSE also allows for the inclusion of additional stochastic terms [21]. However, in common master equations for Brownian motion, the noise cancels if the system is in a coherent state. For the problems being discussed in this paper, it is sufficient to consider the Gaussian WP solutions of our NLSE (7) that exist for the damped free motion and the damped harmonic oscillator, as in the undamped case. They can physically be interpreted as being similar to coherent states; in cases where they represent minimum uncertainty WPs, they even fulfil exactly one of the definitions of coherent states [11]. In order to examine the influence of the environment on these WPs, the dynamics and energetics of Gaussian WPs shall be written in an appropriate form, as shown in the following section. The relations will be given explicitly for the harmonic oscillator, $V = \frac{m}{2}\omega^2 x^2$. If not otherwise specified, the corresponding expressions for the free motion, $V = 0$, can be arrived at by taking the limit $\omega \to 0$.

3. Dynamics and energetics of Gaussian wave packets

For both the problems—the free motion and harmonic oscillator—addressed in this paper, exact analytical solutions exist for the conservative SE, as well as for our dissipative NLSE. In particular, Gaussian WP-type solutions can be found in all these cases [32]. The particle aspect of the system is expressed by the maximum of the WP and its time evolution, the wave aspect by the width of the WP and its capability of spreading in time.

Writing the WP solution of the SE explicitly in the form

$$\Psi_{\rm WP}(x,t) = N(t)\exp\left\{ i\left[y(t)\tilde{x}^2 + \frac{\langle p\rangle}{\hbar}\tilde{x} + K(t)\right]\right\} \tag{8}$$

(where $\tilde{x} = x - \langle x\rangle = x - \eta(t)$), shows that the maximum at position $\langle x\rangle$ follows the classical trajectory $\eta(t)$. The explicit form of $N(t)$ and $K(t)$ is not relevant to the following discussion. The WP-width, $\sqrt{\langle\tilde{x}^2\rangle}$ (where $\langle\tilde{x}^2\rangle = \langle x^2\rangle - \langle x\rangle^2$), is connected with the imaginary part of the complex coefficient of $\tilde{x}^2$ in the exponent, $y(t) = y_{\rm R}(t) + i y_{\rm I}(t)$, via

$$\frac{2\hbar}{m} y_{\rm I} = \frac{\hbar}{2m\langle\tilde{x}^2\rangle} = \frac{1}{\alpha^2(t)}. \tag{9}$$

Inserting the WP into the SE proves that $\langle x\rangle = \eta(t)$ obeys the classical Newtonian equation for a corresponding point particle,

$$\ddot{\eta} + \omega^2\eta = 0. \tag{10}$$

To determine the time-dependence of the WP-width, the complex, quadratically nonlinear equation of Riccati type,

$$\frac{2\hbar}{m}\dot{y} + \left(\frac{2\hbar}{m}y\right)^2 + \omega^2 = 0 \tag{11}$$

must be solved. Using

$$\frac{2\hbar}{m} y_{\rm R} = -\frac{1}{2}\frac{\dot{y}_{\rm I}}{y_{\rm I}} = \frac{\dot{\alpha}}{\alpha} \tag{12}$$

this complex equation can also be transformed into the real nonlinear Newton-type equation

$$\ddot{\alpha} + \omega^2\alpha = \frac{1}{\alpha^3} \tag{13}$$

for the variable $\alpha = (2m\langle\tilde{x}^2\rangle/\hbar)^{1/2}$ which was introduced in equation (9) and is, apart from a constant factor, identical with the WP-width. This variable will be useful later on to express other physical quantities, such as ground-state energies. Equation (13) was first discussed by

Ermakov (1880!) [33] and later reconsidered in a physical, particularly quantum mechanical, context by several other authors [34]. The solution of (13) also provides information on the non-classical part of the dynamics (and, thus, the tunnelling currents) contained in the convection current density that occurs in the continuity equation (2) and the Fokker–Planck equation (4), since the corresponding velocity field v_- can be obtained from α and $\dot{\alpha}$ as

$$v_- = \frac{2\hbar}{m} y_R \tilde{x} + \langle v \rangle = \frac{\dot{\alpha}}{\alpha}\tilde{x} + \dot{\eta}. \tag{14}$$

Particle and wave aspects of the quantum systems being considered are not only reflected by the occurrence of two different equations of motion describing the dynamics of the system but, also, the energy contains two corresponding contributions. Calculating the energy mean value with the Gaussian WP yields

$$\begin{aligned} \langle E \rangle = \langle H_L \rangle &= \frac{1}{2m}\langle p^2 \rangle + \frac{m}{2}\omega^2 \langle x^2 \rangle \\ &= \left(\frac{1}{2m}\langle p \rangle^2 + \frac{m}{2}\omega^2 \langle x \rangle^2 \right) + \left(\frac{1}{2m}\langle \tilde{p}^2 \rangle + \frac{m}{2}\omega^2 \langle \tilde{x}^2 \rangle \right) \\ &= E_{cl} + \tilde{E}_L. \end{aligned} \tag{15}$$

This is different from the classical energy, E_{cl}, due to the dispersion of position and momentum. The energy difference $\tilde{E}_L$ can also be expressed in terms of the real and imaginary parts of the complex variable $y(t)$, or by the variable α and its time-derivative,

$$\tilde{E}_L = \frac{\hbar^2}{2m}\frac{y_R^2 + y_I^2}{y_I} + \frac{m}{2}\omega^2 \frac{1}{y_I} = \frac{\hbar}{4}\left(\dot{\alpha}^2 + \frac{1}{\alpha^2} \right) + \frac{\hbar}{4}\omega^2 \alpha^2. \tag{16}$$

This quantity is just the constant ground-state energy and can even be written as a Hamiltonian that provides the equation of motion for α, provided appropriate canonical momenta are introduced (for details see [35]).

So, the dynamics (WP-spreading and tunnelling currents), as well as the energetics (ground-state energy), of the quantum mechanical wave aspect of Gaussian WPs can be obtained from the solutions of the Riccati equation (11) for $y(t)$ (since $\alpha^2 \propto 1/y_I$), or directly from the solutions of the Ermakov equation (13) for $\alpha(t)$. Qualitative changes in the solutions of these equations will result in qualitative changes of the dynamics and energetics of the quantum mechanical wave aspect. The effect of a dissipative environment on these equations will be discussed in the next section.

4. Changes due to the breaking of time-reversal symmetry

Adding a nonlinear term that takes into account the dissipative environment in our NLSE (7) does not affect the general form (8) of the WP solution; only the equations of motion for η, y and α, as well as the dependence of the ground-state energy $\tilde{E}$ on these quantities, change.

The equation for η describing the motion of the WP-maximum thus fulfils the correct equation of a classical particle with linear velocity-dependent frictional force

$$\ddot{\eta} + \gamma\dot{\eta} + \omega^2 \eta = 0. \tag{17}$$

Likewise, a linear term containing the friction coefficient γ is added to the Riccati equation for $y(t)$,

$$\frac{2\hbar}{m}\dot{y} + \gamma\left(\frac{2\hbar}{m}y\right) + \left(\frac{2\hbar}{m}y\right)^2 + \omega^2 = 0. \tag{18}$$

The relation between the imaginary part, y_I, and α remains unchanged, as given by (9); however, the real part, y_R, now fulfils

$$\frac{2\hbar}{m} y_R = -\frac{1}{2}\frac{\dot{y}_I}{y_I} - \frac{\gamma}{2} = \frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2} \tag{19}$$

thus leading to the modified Ermakov equation

$$\ddot{\alpha} + \left(\omega^2 - \frac{\gamma^2}{4}\right)\alpha = \frac{1}{\alpha^3}. \tag{20}$$

The ground-state energy, $\tilde{E}_{NL}$, expressed in terms of y_R and y_I, remains formally unchanged but, since the relation between y_R and α is different from the conservative case, $\tilde{E}_{NL}$ expressed in terms of α now has the form

$$\tilde{E}_{NL} = \frac{\hbar}{4}\left[\left(\dot{\alpha} - \frac{\gamma}{2}\alpha\right)^2 + \frac{1}{\alpha^2}\right] + \frac{\hbar}{4}\omega^2\alpha^2. \tag{21}$$

As in the case of the classical energy of a frictionally damped dissipative system, also the quantum mechanical contribution $\tilde{E}_{NL}$ is now (generally) no longer constant. The change of the ground-state energy can be expressed in terms of the correlation of position- and momentum-fluctuations as

$$\frac{d}{dt}\tilde{E}_{NL} = -\frac{\gamma}{4}\frac{d}{dt}\langle[\tilde{x}, \tilde{p}]_+\rangle_{NL} \tag{22}$$

where $[\ldots,\ldots]_+$ denotes the anti-commutator. This part of the energy that is transferred to the environment can be expressed in terms of α as

$$\frac{\gamma}{4}\langle[\tilde{x}, \tilde{p}]_+\rangle_{NL} = \gamma\frac{\hbar}{4}\left(\dot{\alpha}\alpha - \frac{\gamma}{2}\alpha^2\right). \tag{23}$$

Therefore, the quantity

$$\tilde{E}_{NL} + \frac{\gamma}{4}\langle[\tilde{x}, \tilde{p}]_+\rangle_{NL} = \frac{\hbar}{4}\left[\dot{\alpha}^2 + \frac{1}{\alpha^2} + \left(\omega^2 - \frac{\gamma^2}{4}\right)\alpha^2\right] = \text{const} \tag{24}$$

is always a constant since it contains both parts of the initial ground-state energy—the one that remains in the system and the one that is exchanged with the environment. Moreover, it can be shown that, with the help of appropriately-defined canonical momenta, this constant of motion can also be used as a Hamiltonian to obtain the equation of motion (20). A similar treatment for the classical degrees of freedom has been shown in [36].

The qualitative effect of the dissipative environment can be discussed most transparently if we consider the modified complex Riccati equation (18). For certain choices of the occurring parameters (e.g., constant ω and γ), the solution of the inhomogeneous Riccati equation can be reduced to the solution of a homogeneous complex Bernoulli equation

$$\frac{2\hbar}{m}\dot{v} + A\left(\frac{2\hbar}{m}v\right) + \left(\frac{2\hbar}{m}v\right)^2 = 0 \tag{25}$$

using the ansatz $\left(\frac{2\hbar}{m}\right)y(t) = \left(\frac{2\hbar}{m}\right)\hat{y} + \left(\frac{2\hbar}{m}\right)v(t)$, where the time-independent particular solution of the Riccati equation can take the values

$$\frac{2\hbar}{m}\hat{y}_\pm = -\left(\frac{\gamma}{2}\right) \pm \left(\frac{\gamma^2}{4} - \omega^2\right)^{1/2}. \tag{26}$$

The coefficient A of the linear term in the Bernoulli equation can also have two different values,

$$A_\pm = 2\left(\frac{\gamma}{2} + \frac{2\hbar}{m}\hat{y}_\pm\right) = \pm 2\left(\frac{\gamma^2}{4} - \omega^2\right)^{1/2} \tag{27}$$

which will be of importance subsequently. Equation (25) can be solved analytically and the solution given in the form

$$\frac{2\hbar}{m}v(t) = \frac{\left(\frac{2\hbar}{m}v_0\right) A\,\mathrm{e}^{-At}}{\left(\frac{2\hbar}{m}v_0\right)(1-\mathrm{e}^{-At}) + A}. \tag{28}$$

From the imaginary parts $\left(\frac{2\hbar}{m}\right)v_{\mathrm{I}}(t)$ and $\left(\frac{2\hbar}{m}\right)\hat{y}_{\mathrm{I}}$, one obtains $\left(\frac{2\hbar}{m}\right)y_{\mathrm{I}} = 1/\alpha^2$, and, thus, the solution of the corresponding Ermakov equation (20). The consequences of the occurrence of the environmental parameter γ in the above-mentioned equations for the dynamics and energetics, shall now be discussed separately for the damped free motion and the damped harmonic oscillator.

4.1. Damped free motion

The *classical* part of the *dynamics*, i.e. the motion of the *WP-maximum*, is affected by the dissipative environment as is expected. Whereas the isolated system moves with constant velocity $\langle v\rangle_L = \dot{\eta}_L = v_0 = \mathrm{const}$, the damped system is *slowed down* by the interaction according to $\langle v\rangle_{\mathrm{NL}} = \dot{\eta}_{\mathrm{NL}} = v_0\exp(-\gamma t)$. Therefore, the position is no longer growing proportional to time t, but approaches a finite value $\langle x\rangle_{\mathrm{NL}}(t\to\infty) = v_0/\gamma$, according to $\langle x\rangle_{\mathrm{NL}} = \eta_{\mathrm{NL}} = \frac{v_0}{\gamma}(1-\exp(-\gamma t))$.

Considering the *time-dependence* of the *WP-width*, the parameter values $\omega = 0$ and $\gamma = 0$ yield, for the isolated system, the particular solution $\frac{2\hbar}{m}\hat{y}_\pm = \pm 0$ and, in the Bernoulli equation, $A = \pm 0$. Therefore, one obtains only one solution that leads to the well-known position uncertainty that is growing proportional to t^2,

$$\langle\tilde{x}^2\rangle_L = \langle\tilde{x}^2\rangle_0\left\{1+(\beta_0 t)^2\right\} \tag{29}$$

where $\beta_0 = \left(\hbar/2m\langle\tilde{x}^2\rangle_0\right)$.

Including the interaction with the environment, i.e. for $\gamma\neq 0$, the particular solution $\frac{2\hbar}{m}\hat{y}$, as well as the parameter A in the Bernoulli equation, can possess *two different* values, thus leading to two WPs with different time-dependence of the spreading of the width. For $A_\pm = \pm\gamma$, one obtains

$$\langle\tilde{x}^2\rangle_+ = \langle\tilde{x}^2\rangle_0\left\{\mathrm{e}^{\gamma t} + \left(\frac{\beta_0}{\gamma/2}\right)^2\sinh^2\frac{\gamma}{2}t\right\} \tag{30}$$

$$\langle\tilde{x}^2\rangle_- = \langle\tilde{x}^2\rangle_0\left\{\mathrm{e}^{-\gamma t} + \left(\frac{\beta_0}{\gamma/2}\right)^2\sinh^2\frac{\gamma}{2}t\right\}. \tag{31}$$

Already, because of the $\sinh^2$-term, both WPs are *spreading faster* than in the isolated case; for $\langle\tilde{x}^2\rangle_+$, due to the $\mathrm{e}^{\gamma t}$-term, even exponentially. A formal mathematical explanation for this behaviour can also be given (see, e.g., [22]). The *energies*, $\tilde{E}_\pm$, calculated with the help of the two solutions of the Riccati equation, make it even more obvious that the two WPs have different physical properties, since

$$\tilde{E}_+ = \frac{\hbar}{4}\beta_0\,\mathrm{e}^{-\gamma t} \tag{32}$$

$$\tilde{E}_- = \frac{\hbar}{4}\beta_0\left[1+\left(\frac{\gamma}{\beta_0}\right)^2\right]\mathrm{e}^{-\gamma t}. \tag{33}$$

The two systems have, for all times, different ground-state energies and the energy difference

$$\Delta\tilde{E} = \tilde{E}_- - \tilde{E}_+ = \frac{\hbar}{4}\frac{\gamma^2}{\beta_0}\,\mathrm{e}^{-\gamma t} = \frac{m}{2}\gamma^2\langle\tilde{x}^2\rangle_0\,\mathrm{e}^{-\gamma t} \tag{34}$$

does not depend on $\hbar$, but only on the parameters of the system $\left(m, \langle\tilde{x}^2\rangle_0\right)$ and, particularly, of the environment (γ). The energy difference disappears for $\gamma \to 0$.

The *energy* contribution from the motion of the *WP-maximum* is, in the conservative as well as in the dissipative case, the same as for a corresponding classical particle and is therefore not relevant to our study.

4.2. The damped harmonic oscillator

Considering, first, the *dynamics* in the *reversible* case, i.e. $\gamma = 0$, the maximum of the WP oscillates with constant frequency ω and constant amplitude, just as a corresponding classical particle. Regarding the WP-width, it is obvious that the particular time-independent solutions $\frac{2\hbar}{m}\hat{y}_\pm = \pm\mathrm{i}\omega$ are purely imaginary, where only the positive solution is physically reasonable and the negative one leads to a divergent WP. Since the WP-width $\langle\tilde{x}^2\rangle_L = \hbar/2m\omega$ is constant, α_L is also constant; thus, because $\dot{\alpha}_L = 0$, there is no contribution to the probability flux via v_-. Oscillatory solutions also exist but are irrelevant to the following discussion.

In the damped *irreversible* case, i.e. $\gamma \neq 0$, one has to distinguish between different relations between the parameters ω and γ. It emerges that the *aperiodic limit*, i.e. $\omega = \gamma/2$. yields the desired qualitative effects. Nevertheless, the case of *undercritical damping*, i.e. $\omega > \gamma/2$, will be mentioned briefly. In this case, the *WP-maximum*, like the classical counterpart, oscillates with reduced frequency $\Omega = (\omega^2 - \gamma^2/4)^{1/2}$ and exponentially-damped amplitude. There is also a solution with constant *WP-width* $\langle\tilde{x}^2\rangle_{\mathrm{NL}} = \hbar/2m\Omega$ that does not contribute to the probability flux. For the oscillatory solutions, the above-mentioned applies.

In the *aperiodic limit*, the *WP-maximum* again displays the classical behaviour, i.e. no oscillations but exponential damping of position and velocity according to $\langle x\rangle_{ap} = \eta_{ap} = v_0 t\exp(-\gamma t/2)$, $\langle v\rangle_{ap} = \dot{\eta}_{ap} = v_0(1 - \gamma t/2)\exp(-\gamma t/2)$. More interesting, however, is the behaviour of the *WP-width* since it is no longer constant but grows according to

$$\langle\tilde{x}^2\rangle_{ap} = \langle\tilde{x}^2\rangle_0\left\{1 + (\beta_0 t)^2\right\} \tag{35}$$

i.e., it grows quadratically in time, as in the case of the free motion without damping. This appears as if the effect of the environment compensates for the effect of the confining harmonic potential. Since $\langle\tilde{x}^2\rangle_{ap} = (\hbar/2m)\alpha_{ap}^2$ is now time-dependent, $\dot{\alpha}_{ap} \neq 0$ and, therefore, there is a contribution to the flux that is identical to the case of the undamped free motion.

What is the environmental effect on the *energetics* in the above-mentioned cases? For the *undercritical damping*, the contribution from the motion of the *WP-maximum* shows, again, the same behaviour as the classical system, namely, an exponential decrease. The *quantum contribution* $\tilde{E}_{\mathrm{NL}}$ shows only a shift in the constant ground-state energy. Compared with the undamped case, where

$$\tilde{E}_L = \frac{\hbar}{2}\omega \tag{36}$$

is valid, the ground-state energy of the undercritically damped harmonic oscillator is slightly higher,

$$\tilde{E}_{\mathrm{NL}} = \frac{\hbar}{2}\omega\left(\frac{\omega}{\Omega}\right) > \tilde{E}_L \tag{37}$$

due to some energy back-transfer from the environment (for further details see [21]).

For the *aperiodic limit*, $\omega = \gamma/2$, the *classical* part of the *energy* also decreases exponentially according to

$$E_{ap} = E_0 \left[\left(1 - \frac{\gamma}{2}t\right)^2 + \left(\frac{\gamma}{2}t\right)^2 \right] \mathrm{e}^{-\gamma t}. \tag{38}$$

The *quantum mechanical* contribution $\tilde{E}_{ap}$, however, is no longer constant, as in the undercritical case, but fulfils

$$\tilde{E}_{ap} = \frac{\hbar}{4}\beta_0 \left[2\left(\frac{\frac{\gamma}{2}}{\beta_0}\right)^2 + \left(1 - \frac{\gamma}{2}t\right)^2 + \left(\frac{\gamma}{2}t\right)^2 \right]. \tag{39}$$

Apart from a constant contribution, it also contains two terms that are growing quadratically in time, such as the terms in the classical energy, but without the exponential damping factor. The initial ground-state energy $\tilde{E}_{ap,0} = \frac{\hbar}{4}\beta_0\left[1 + 2\left(\frac{\frac{\gamma}{2}}{\beta_0}\right)^2\right]$ drops after $t_{\min} = 1/\gamma$ to its minimum value $\tilde{E}_{ap,\min} = \frac{\hbar}{4}\beta_0\left[\frac{1}{2} + 2\left(\frac{\frac{\gamma}{2}}{\beta_0}\right)^2\right]$ and, afterwards, grows quadratically in time. Where does the energy gained by the quantum system come from? Considering the term describing the correlations of position- and momentum-fluctuations,

$$\frac{\gamma}{4}\langle[\tilde{x}, \tilde{p}]_+\rangle_{ap} = -\frac{\hbar}{4}\beta_0 \left[\left(1 - \frac{\gamma}{2}t\right)^2 + \left(\frac{\gamma}{2}t\right)^2 \right] + \frac{\hbar}{4}\beta_0 \left[1 - 2\left(\frac{\frac{\gamma}{2}}{\beta_0}\right)^2 \right] \tag{40}$$

that represents the effect of the environment, one finds the same terms growing quadratically in time, as in $\tilde{E}_{ap}$, only with a negative sign; and the sum of quantum energy contributions of the system and the environment,

$$\tilde{E}_{ap} + \frac{\gamma}{4}\langle[\tilde{x}, \tilde{p}]_+\rangle_{ap} = \frac{\hbar}{4}\beta_0 = \text{const} \tag{41}$$

is always a constant, as already mentioned above.

5. Conclusions

We found that, according to our model, the interaction with a dissipative environment should lead to new qualitative effects on the dynamics and energetics of quantum systems. The breaking of time-reversal symmetry introduces a new (environmental) parameter, the friction coefficient γ, with several consequences.

The occurrence of a nonzero parameter γ creates the possibility of bifurcations in the Riccati equation (18). For the damped free motion, this leads to two different WP-solutions with different quantum dynamics, i.e. different spreading behaviour of the WP-width. In both cases, however, the WPs of the interacting quantum systems are spreading faster than the WP of the isolated system. This shows some similarity with experimental results by Nimtz [37], where Gaussian-like pulses reach a higher signal velocity in the presence of barriers—i.e. obstacles, such as collision partners of some environment—than without. Any connection between our theoretically-predicted and this experimental effect will be investigated in a forthcoming work. Also, the tunnelling behaviour, according to the $\dot{\alpha}/\alpha$-term in the current of the continuity equation/Fokker–Planck equation, is affected by this deviant spreading behaviour. When considering the energies of the WPs, it also becomes obvious that the two WP-solutions actually describe systems that are physically distinguishable. The breaking of the time-reversal symmetry leads to a splitting of the ground-state energy—similar to the splitting of energy levels due to reduction of spatial symmetry—where the energy gap between

$\tilde{E}_+$ and $\tilde{E}_-$ is dependent on the parameter γ and disappears for $\gamma \to 0$. Although both ground-state energies decay exponentially, they are different for all times. In principle, this energy gap should be measurable, but a proper experimental set-up has not yet been found.

In the case of the damped harmonic oscillator, the environmental parameter γ presents the possibility of compensating for the effect of the harmonic potential, represented by the parameter ω, the oscillation frequency, if $\omega = \gamma/2$ is fulfilled. In this aperiodic limit, the dynamics of the quantum system shows a qualitatively different behaviour from the undercritical damped situation, $\omega > \gamma/2$. This behaviour is similar to a resonance phenomenon where, as long as the oscillatory frequency of the system and the frequency of the external excitation differ, there is no effect. However, as soon as these two frequencies are identical, absorption of energy, i.e. a flux of energy from the external excitation to the system, occurs. In our case, the external frequency is (half of) the damping parameter (collision frequency) γ. For $\gamma/2 = \omega$, a flux is created that does not exist without the environment, and energy is transferred from it to the system.

Considering the environment as a heat bath (eventually infinite), this might appear, at first sight, to be a violation of the laws of thermodynamics. However, this is not the case since the energy from the heat bath is not transferred into the classical degrees of freedom of the system; in other words, the WP does not start oscillating with increasing amplitude in this resonance-like situation, but is still damped exponentially. Only the quantum part of the system's energy, $\tilde{E}$, absorbs energy from the surroundings, so thermal energy is transferred into quantum mechanical energy, and, furthermore, in a way that the sum of the energies of system and surroundings always remains constant. Somehow, this displays similarities with thermal chemical reactions, but it would be desirable to find experiments in which this resonance-like behaviour of the flux can also be detected. The macroscopic quantum effects mentioned in the introduction, where the variation of an external parameter, the temperature T (which is usually proportional to the collision frequency), also under resonance-like conditions, leads to the creation of fluxes (quantum-Hall flux, superconducting current), show some qualitative similarities. However, the questions of whether there are any connections with our above-mentioned results and of what kind they might be require further investigation.

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